Characterization of algebraic polynomial of best uniform approximation

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We are going to solve the problem of approximation in the case of uniform approximation by means of algebraic polynomials. This problem was solved by P.L. Chebyshev in the last century with his famous alternation theorem. The problem may be stated as follows:
Let $f \in C[a,b]$ and let $E_n(f) := inf_{q\in\prod_n} \|f-q\|_{C[a,b]}$. Find $p \in \prod_n$ such that
$E_n(f) = \|f-p\|_{C[a,b]}$, where $\prod_n$ denotes the set of all polynomials of degree n.

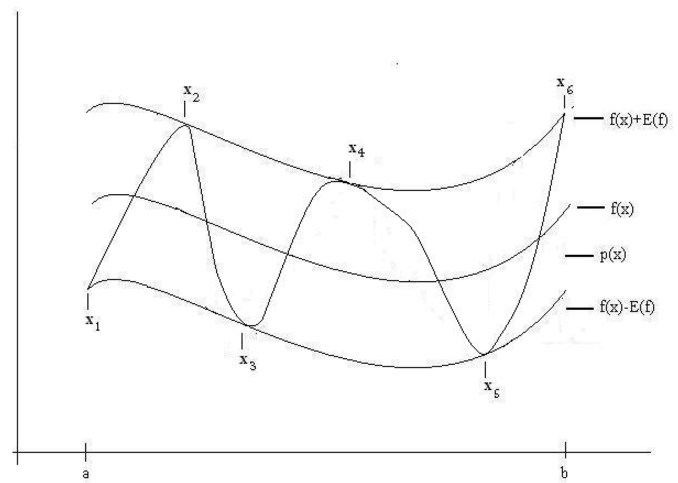
In what follows we shall use $\|\cdot\|$ in place of $\|\cdot\|_{C[a,b]}$ for convenience.

Definition 1. Let $f \in C[a,b]$. The polynomial $p \in \prod_n$ is said to realize Chebyshev alternation for f in $[a,b]$ if there exist n+2 points x_i, $i = 1,...n+2$, $a \le x_1 < x_2... < x_{n+2} \le b$ such that

$$f(x_i) - p(x_i) = \epsilon(-1)^i \|f-p\|_{C[a,b]}$$

where the number ϵ is $+1$ or -1.

The Chebyshev alternation has the following geometric interpretation: if the polynomial $p \in \prod_n$ is said to realize Chebyshev alternation for $f \in C[a,b]$ in $[a,b]$, then p lies between the two vertical translations of f, $\varphi(x) = f(x)+\|f-p\|$ and $\psi(x) = f(x)-\|f-p\|$. Moreover, p alternately touches φ and ψ at least n+2 times. An illustration of Chebyshev alternation for the case $n = 4$ is given below.



Theorem 2. *(Chebyshev alternation theorem): Let $f \in C[a,b]$. The algebraic polynomial $p \in \prod_n$ is the polynomial of best uniform approximation in $\prod_n$ for f if and only if p realizes Chebyshev alternation for f in $[a,b]$*

Proof. First let $p \in \prod_n$ realize Chebyshev alternation for f in $[a,b]$ and let $x_j, j = 1,...,n+2$ be the points of as defined in the definition for Chebyshev alternation. Let's assume that p is not a polynomial of best uniform approximation , but $q \in \prod_n$ is. Then

$$E_n(f) = \|f-q\| < \|f-p\|$$

From the above inequality we get

$$|f(x_j) - q(x_j)| < |f(x_j) - p(x_j)|, j = 1,..,n+2$$

since $|f(x_j) - p(x_j)| = \|f - p\|$. Then from here we see that

$$\begin{aligned} sign(p(x_j) - q(x_j)) &= sign(p(x_j) - f(x_j) + f(x_j) - q(x_j)) \\ &= sign(p(x_j) - f(x_j)). \end{aligned}$$

So $p-q$, which is in $\prod_n$, takes the sign of $p-f$. Since these functions are continuous, this also tells us that $p-q$ has the same zeros as $p-f$. Since p realizes Chebyshev alternation for f in $[a,b]$, from the Intermediate Value Theorem we know $p-f$ has at least $n+1$ zeros. So $p-q$ has at least $n+1$ zeros, which is possible only if $p-q \equiv 0$, i.e. $p=q$, which is a contradiction to the assumption.

Now suppose we have $p \in \prod_n$ that is of best uniform approximation of f from $\prod_n$. We would like to show that p realizes Chebyshev alternation for f. So assume to the contrary that $m+2$ is the highest number of points $x_1 < x_2 < ... < x_{m+2}$ in $[a,b]$ such that

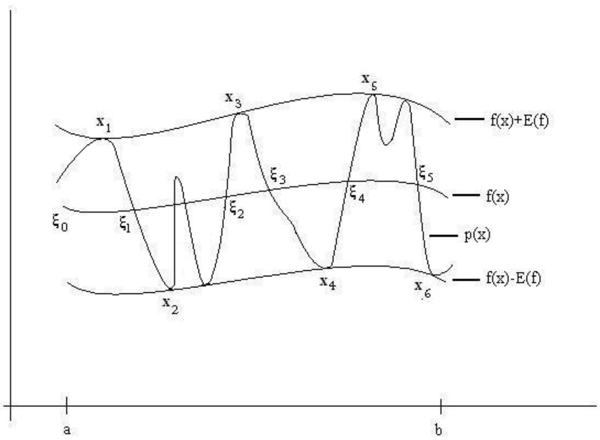
$$f(x_i) - p(x_i) = \epsilon(-1)^i \|f-p\|, \ i = 1, ..., m+2, \tag{1}$$

where ϵ is $+1$ or -1 and m<n. We need m to be nonegative in order to continue the proof. If m was negative (-1 is the only possiblity), then we have just one point, say $x_0 \in [a,b]$ so that $|f(x_0) - p(x_0)| = \|f-p\|$ is achieved. Then from here it is clear that a better uniform approximation can be achieved in $\prod_n$ simply by a vertical shift of p. So $m \geq 0$.

We will just suppose that $\epsilon = 1$, and the proof is similar otherwise. From the assumption about the alternation of $p-f$ on the intervals $[x_i, x_{i+1}]$ and by the Intermediate Value Theorem, there exist points $\xi_0, \xi_1, ..., \xi_{m+1}$, which satsify $a = \xi_0 \leq x_1 < \xi_1 < x_2 < \xi_2 < ... < \xi_{m+1} < x_{m+2} \leq b$ and $f(\xi_i) = p(\xi_i)$, for $i = 1, ..., m+1$. Moreover, we may choose ξ_i such that for every $x \in [\xi_{i-1}, \xi_i]$ we have

$$(-1)^i (f(x) - p(x)) > -E_n(f), \ i = 1..., m+1. \tag{2}$$

This is illustrated in the figure below for the case $m = 4$.



Notice in the figure that simply to satisfy $f(\xi_i) = p(\xi_i)$, we are left with 3 different choices for ξ_2, but in order to satisify (2) we simply choose the largest point ξ_2 between x_2 and x_3 such that $f(\xi_i) = p(\xi_i)$. In general, we are able to satisfy (2) in a similar way.

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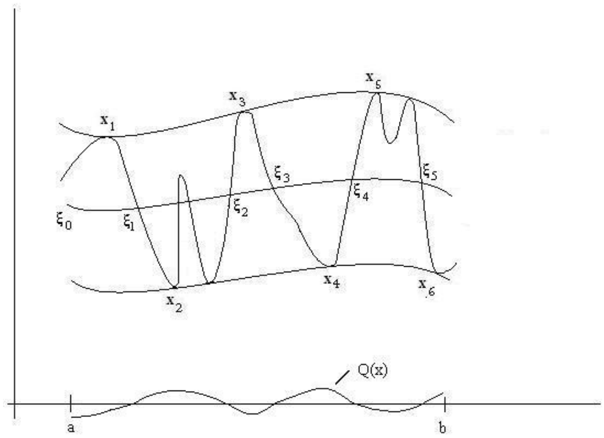
The main idea from here is to construct a polynomial $Q(x) \in \prod_{m+1}$ so that $Q(x) + p(x)$, which is obviously in $\prod_n$, is of better uniform approximation to $f(x)$ than $p(x)$. The idea is to build the interpolation polynomial $Q(x)$ that interpolates the points $(\xi_i, 0)$, $i = 1, ...m+1$ so that for $x \in [\xi_{i-1}, \xi_i]$, $Q(x) - f(x)$ takes the sign opposite of $p(x_i) - f(x_i)$, and $\|Q(x)\|$ is very small. If this is done, then one can see from the illustration below, that $p(x) + Q(x)$ will squeeze inside of what was assumed to be $f(x) + E_n(f)$ and $f(x) - E_n(f)$, thus creating a polynomial of better approximation. The details of this construction are all that remains in the proof. A possible polynomial $Q(x)$ to do this job for the same example that was given above is illustrated here.



Now, since (2) holds on the compact interval $[\xi_{i-1}, \xi_i]$ for each i, there exists δ so that

$$(-1)^i(f(x) - p(x)) > \delta - E_n(f), \text{ for } x \in [\xi_{i-1}, \xi_i], i = 1..., m+1. \tag{3}$$

Let us set

$$Q(x) = (-1)^{m+1}\lambda(x - \xi_1)\cdots(x - \xi_{m+1}),$$

where

$$\lambda = \frac{\delta}{2(b-a)^{m+1}}.$$

Note that $Q \in \prod_n$. Now we would like to show that $p + Q$ is of better uniform approximation to f than p.

From the definition of Q it follows that

$$|Q(x)| \leq |(-1)^i\delta/2| = \delta/2 \text{ for } x \in [a, b] \tag{4}$$

Moreover, we can see verify that for $x \in [\xi_{i-1}, \xi_i]$ we have

$$0 < (-1)^i Q(x) \tag{5}$$

Now combining (1) – (5), on $[\xi_{i-1}, \xi_i]$ we have the following:

$$\begin{aligned}(-1)^i(f(x) - p(x) - Q(x)) &> \delta - E_n(f) - \delta/2 \\ &= \delta/2 - E_n(f)\end{aligned}$$

And obviously $f(\xi_i) - p(\xi_i) - Q(\xi_i) = 0$, for $i = 1, ..., m+1$. And so it follow that

$$|f(x) - p(x) - Q(x)| < E_n(f) - \delta/2$$

everywhere on $[a, b]$, so that we have $\|f - p - Q\| < \|f - p\|$, i.e. a contradiction, since $p + Q \in \prod_n$.

□

Theorem 3. *Let $f \in C[a, b]$. For every natural number n there exists a unique algrebraic polynomial $p \in \prod_n$ of best uniform approximation to f in $\prod_n$.*

Proof. Let $p \in \prod_n$ and $q \in \prod_n$ be two algebraic polynomials of best uniform approximation to f, i.e.

$$\|f - p\| = \|f - q\| = E_n(f) \tag{6}$$

Then we know that any convex combination of p and q is also of best uniform approximation. So $g = (p+q)/2 \in \prod_n$ is of best uniform approximation to f. So by the Chebyshev alternation theorem, g realizes Chebyshev alternation for f, i.e there exists n+2 points $x_i,\ i = 1, ..., n+2,\ a \le x_1 < x_2 \cdots < x_{n+2} \le b$, such that

$$f(x_i) - (p(x_i) + q(x_i))/2 = \epsilon(-1)^i E_n(f),\ i = 1, ..., n+2, \tag{7}$$

where $\epsilon = 1$ or -1.

So then from (7) we then have

$$E_n(f) = |f(x_i) - p(x_i)/2 - q(x_i)/2| \le |f(x_i)/2 - p(x_i)/2| + |f(x_i)/2 - q(x_i)/2| \tag{8}$$

for $i = 1, ..., n+2$. And from (6) we know that $|f(x_i) - p(x_i)| \le E_n(f)$ and $|f(x_i) - q(x_i)| \le E_n(f)$. Thus (7) is possible only if $f(x_i) - p(x_i) = f(x_i) - q(x_i)$, i.e. $p(x_i) = q(x_i)$ for $i = 1, ...n+2$. Finally, since $p, q \in \prod_n$ we must have that $p = q$. □

Theorem 4. *Let $f \in C[a, b], p \in \prod_n$ and $x_i,\ i = 1, ..., n+2, a \le x_1 < x_2 < ... < x_n + 2 \le b$, be n+2 different points in $[a, b]$. If the difference $f - p$ has alternate signs at the points $x_i,\ i = 1, ...n+2$, then*

$$E_n(f) \ge \mu = min\{|f(x_i) - p(x_i)| : i = 1, ...n+2\}.$$

Proof. Let us assume that $E_n(f) < \mu$. Let $q \in \prod_n$ be the algebraic polynomial of best uniform approxiamtion to f, i.e. $\|f - q\| = E_n(f) < \mu$. It follows that

$$sign(p(x_i) - q(x_i)) = sign(p(x_i) - f(x_i) + f(x_i) - q(x_i)) = sign(p(x_i) - f(x_i)),$$

i.e. the sign of $p - q$ takes the sign of $p - f$ at the points $x_i, i = 1, ..., n+2$. Therefore since $p - f$ has alternate signs at n+2 points, so does $p - q$. Thus $p - q \in \prod_n$ has n+1 zeros by the Intermediate Value theorem. So $p = q$, contradicting

$$\|f - q\| = E_n(f) < \mu \le \|f - p\|.$$

□